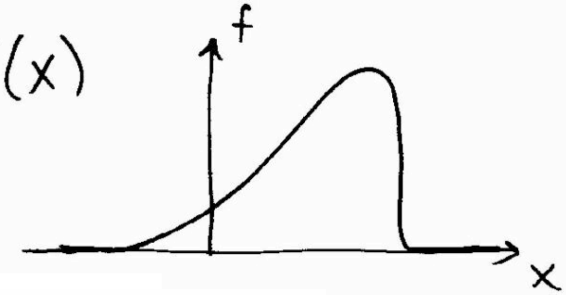
Chap 2 - Waves

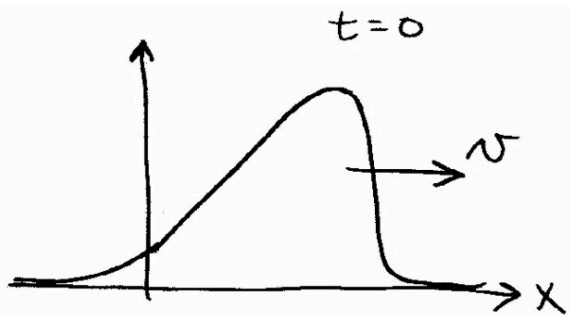
eg. $f(x)$



change variable,

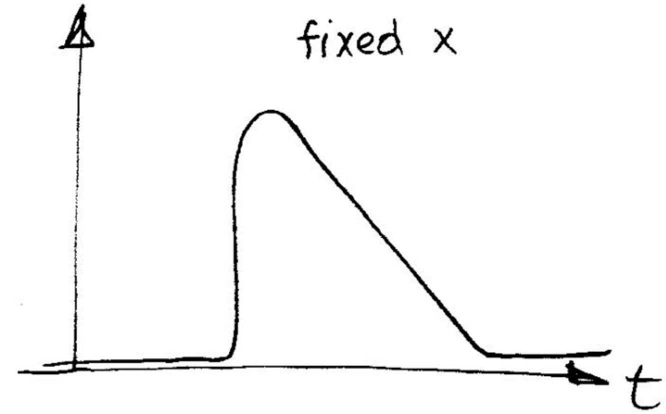
$$x \to x - vt$$

$f(x-vt)$:



moves to right

[$f(x+vt)$ moves to left]



- wave maintains its shape for case considered here (v = const, indep. of frequency component ie. "dispersionless").

wave eqn ($v = \text{const}$):

$$\psi(x,t) = f(x \mp vt) \equiv f(x')$$

$$(x' \equiv x \mp vt)$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x'} \underbrace{\frac{\partial x'}{\partial x}}_{=1} = \frac{\partial f}{\partial x'}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial t} = \mp v \frac{\partial f}{\partial x'}$$

and, $$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 f}{\partial (x')^2}$$

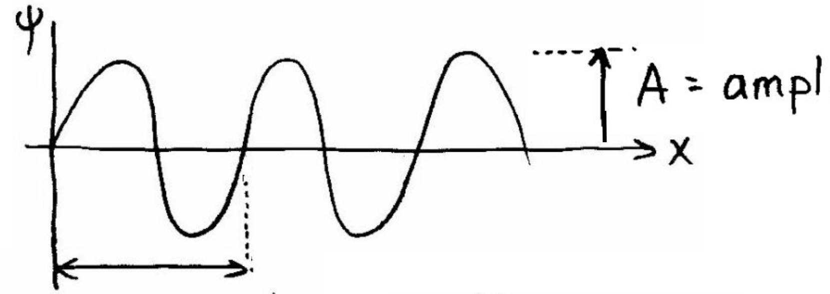
$$\frac{\partial^2 \psi}{\partial t^2} = (\mp v)^2 \frac{\partial^2 f}{(\partial x')^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}}$$

2nd order, homogeneous, undamped...

Harmonic Waves (sines and cosines)

$$\psi(x,t) = A \sin k(x - vt)$$



λ = wavelength (or spatial period) ; $k\lambda = 2\pi$

$$\Rightarrow \boxed{k = \frac{2\pi}{\lambda}}$$

k = propogation number (or "wave number")

T = temporal period (or just "period")

$$kvT = 2\pi$$

$$\Rightarrow T = \frac{2\pi}{kv} = \frac{\lambda}{v}$$

ν = temporal frequency (or "frequency")

$$\equiv \frac{1}{T} \quad (s^{-1}) \quad \Rightarrow \quad \boxed{v = \nu\lambda}$$

also, ω = angular temporal freq.

$$\omega = 2\pi\nu = \frac{2\pi}{T} \quad \left(\frac{\text{rad}}{\text{s}}\right)$$

$$\psi(x,t) = A \sin k(x - vt) = A \sin(kx - \omega t)$$

since $kv = \frac{2\pi}{\lambda}\nu\lambda = 2\pi\nu$

and,

$K = \frac{1}{\lambda}$ (cm^{-1}) = wave number (for chemists)

often used as unit of energy for photons.

$$E = h\nu = \frac{hc}{\lambda} = \frac{2\pi\hbar c}{\lambda} = (2\pi\hbar c)K$$

$$2\pi\hbar c = 2\pi(1973\ \text{eV}\cdot\text{Å}) = \frac{1}{8.07}\ \frac{\text{meV}}{\text{cm}^{-1}}$$

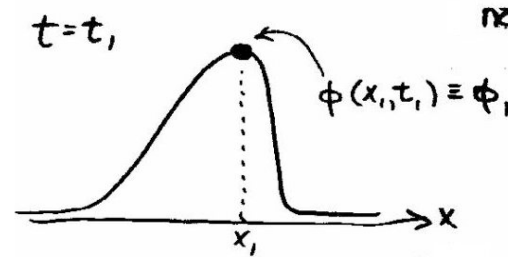
phase:

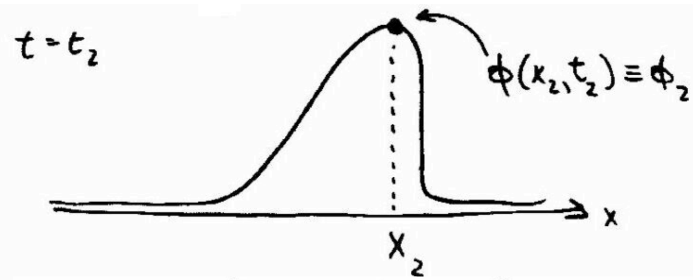
back to, $\psi(x,t) = A \sin(\underbrace{kx - \omega t}_{\equiv\, \phi(x,t)})$ phase

or, with initial phase ε,

$$\phi(x,t) = (kx - \omega t + \varepsilon)$$

also useful for nonharmic waves.



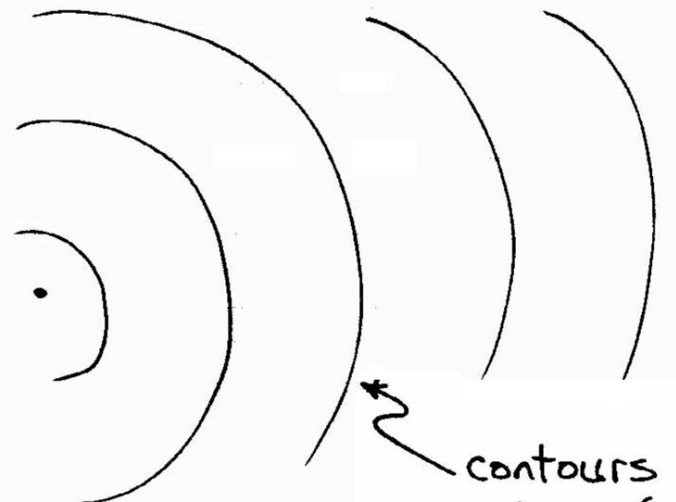


wave is fcn only of $x - vt$ or $kx - \omega t$ $\Rightarrow$ $\psi(x,t) = \tilde{\psi}(\phi)$

$\Rightarrow \phi_1 = \phi_2$ above.

$\therefore$ the phase denotes a specific point on a wave.

eg. circular waves:



speed of propagation of point of fixed phase:

$$x = \frac{1}{k}\phi + \frac{\omega}{k}t - \frac{1}{k}\varepsilon$$

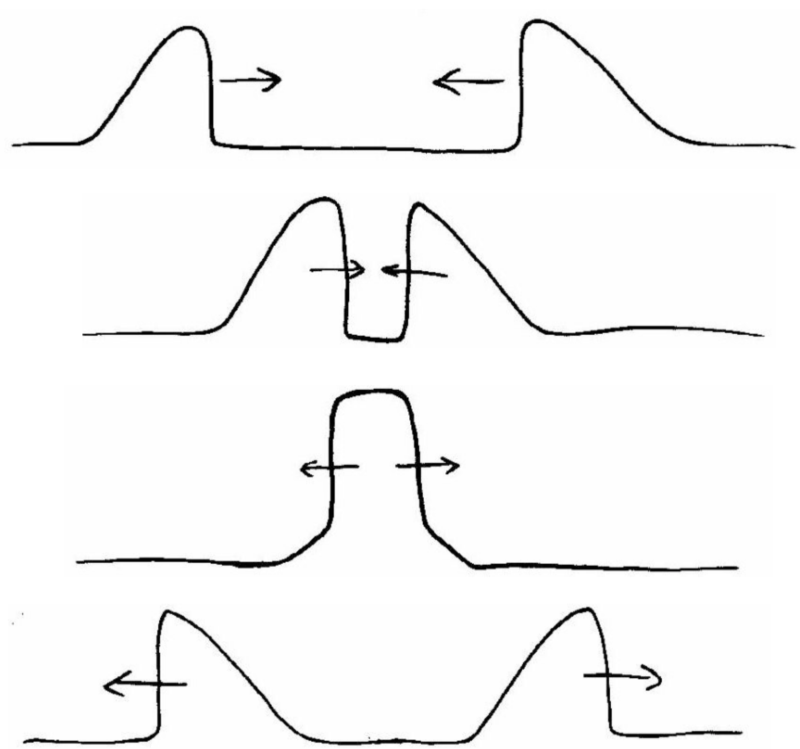
$$\left(\frac{\partial x}{\partial t}\right)_\phi = \frac{\omega}{k} = v \qquad \text{phase velocity}$$

Superposition:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{is linear in } \psi$$

$\Rightarrow$ if ψ_1 and ψ_2 are solns, then $\psi_1 + \psi_2$ is also a soln.

ie. waves do not interact !



Complex Representation:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

eg. $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

use for waves:

$$\psi(x,t) = A\cos(kx-\omega t+\varepsilon) = \mathrm{Re}\left[Ae^{i(kx-\omega t+\varepsilon)}\right]$$

often just write as:

$$\psi(x,t) = Ae^{i(kx-\omega t+\varepsilon)}$$

with Re[...] understood.

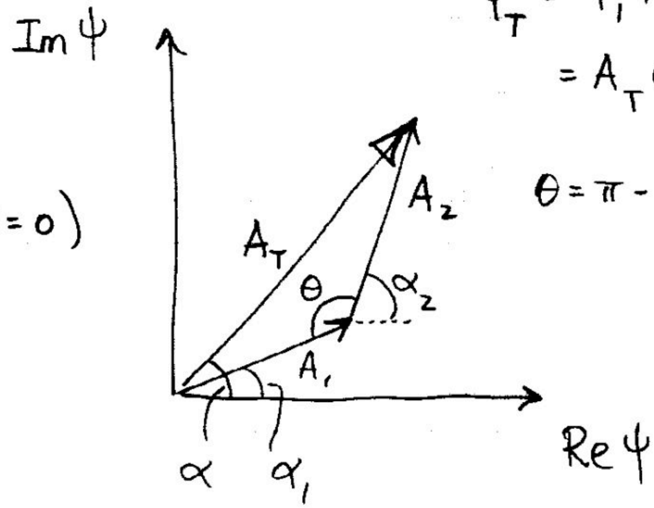
Phasors: add waves of same freq. at fixed point in space.

eg. $\psi_1 = A_1 e^{i(\alpha_1-\omega t)}$; $\alpha_1 = kx_1 + \varepsilon_1$

$\psi_2 = A_2 e^{i(\alpha_2-\omega t)}$; $\alpha_2 = kx_2 + \varepsilon_2$

$$\psi_T = \psi_1 + \psi_2 = A_T e^{i(\alpha-\omega t)}$$

$(t=0)$



$$\theta = \pi - (\alpha_2-\alpha_1)$$

$$A_T^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2-\alpha_1)$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{A_1\sin\alpha_1 + A_2\sin\alpha_2}{A_1\cos\alpha_1 + A_2\cos\alpha_2}$$

3-D Waves

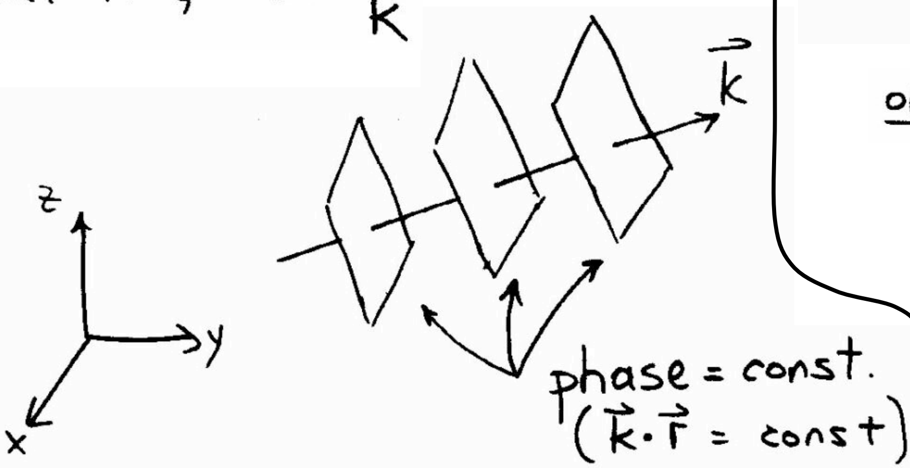
$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

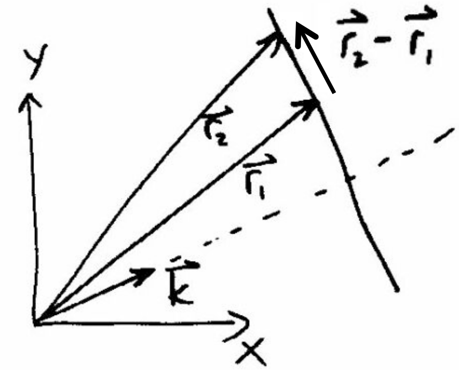
one soln: harmonic plane waves

$$\psi = A\sin(\vec{k}\cdot\vec{r} - \omega t)$$

or
$$\psi = A e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$
$$= A e^{i(k_x x + k_y y + k_z z - \omega t)}$$
$$= A e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t}$$

$$|\vec{k}| = k \quad ; \quad v = \frac{\omega}{k}$$



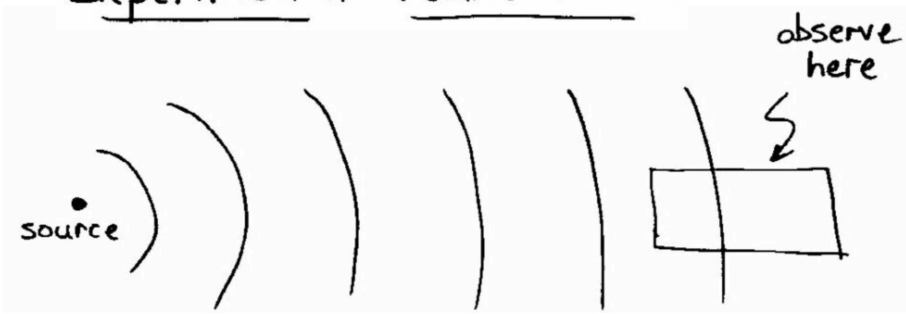


$$\vec{k}\cdot(\vec{r}_2 - \vec{r}_1)$$
$$= \vec{k}\cdot\vec{r}_2 - \vec{k}\cdot\vec{r}_1$$
$$= 0$$

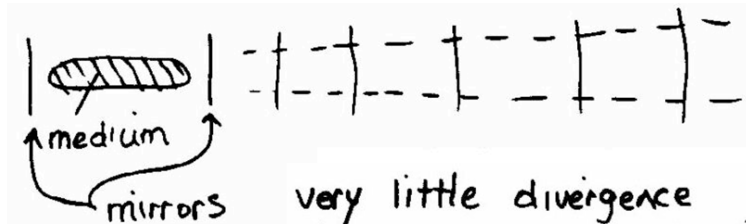
$$\Rightarrow \vec{k} \perp (\vec{r}_2 - \vec{r}_1)$$

- wave-vector is $\perp$ to any vector in wavefront.

Experimental realization?



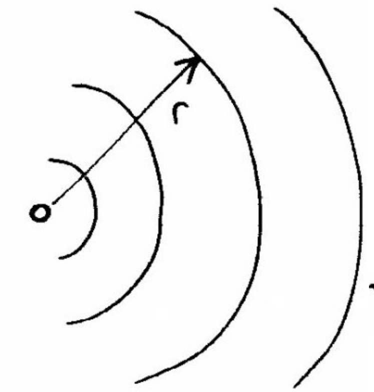
or laser:



very little divergence (amp $\neq$ const on wavefront).

other solns:

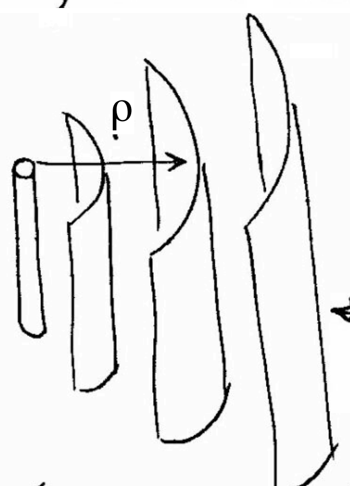
- spherical harmonic waves



$$\psi(r,t) = \left(\frac{A}{r}\right) e^{i(kr-\omega t)}$$

spherical wavefronts
($r = \text{const}$)

- cylindrical harmonic waves



$$\psi(\rho,t) = \left(\frac{A}{\sqrt{\rho}}\right) e^{i(k\rho-\omega t)}$$

cylindrical wavefronts
($\rho = \text{const}$)

(ρ in x,y plane)